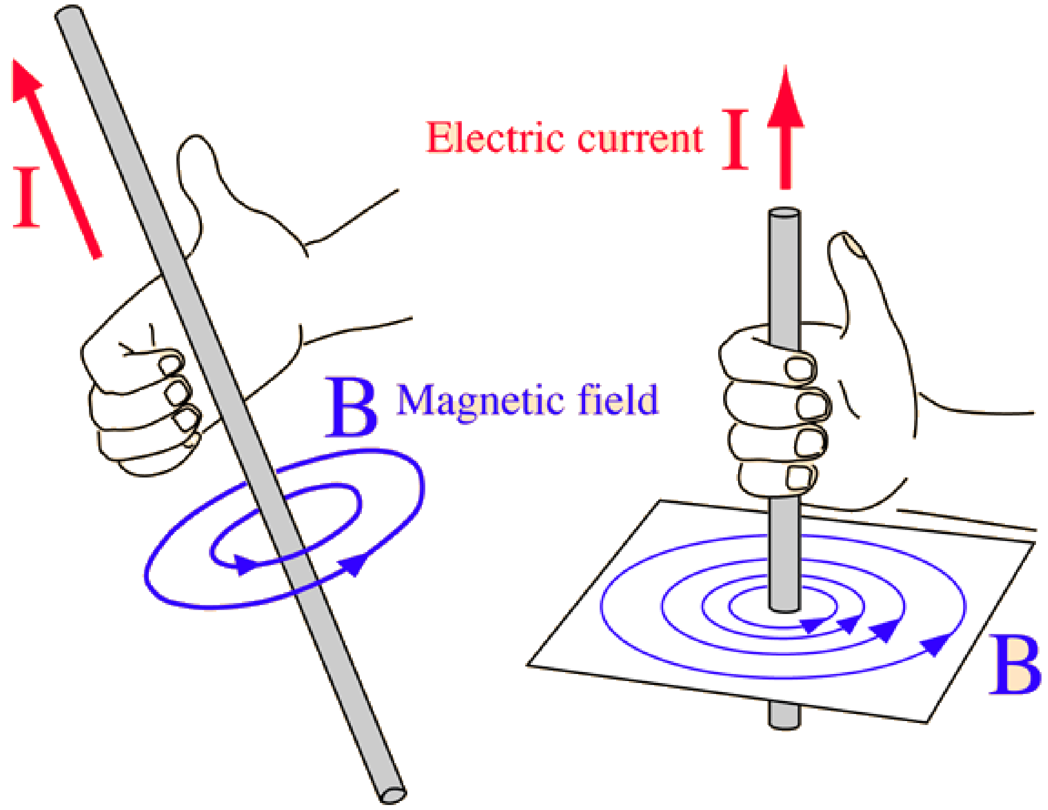
Week 5: Static Magnetic Field

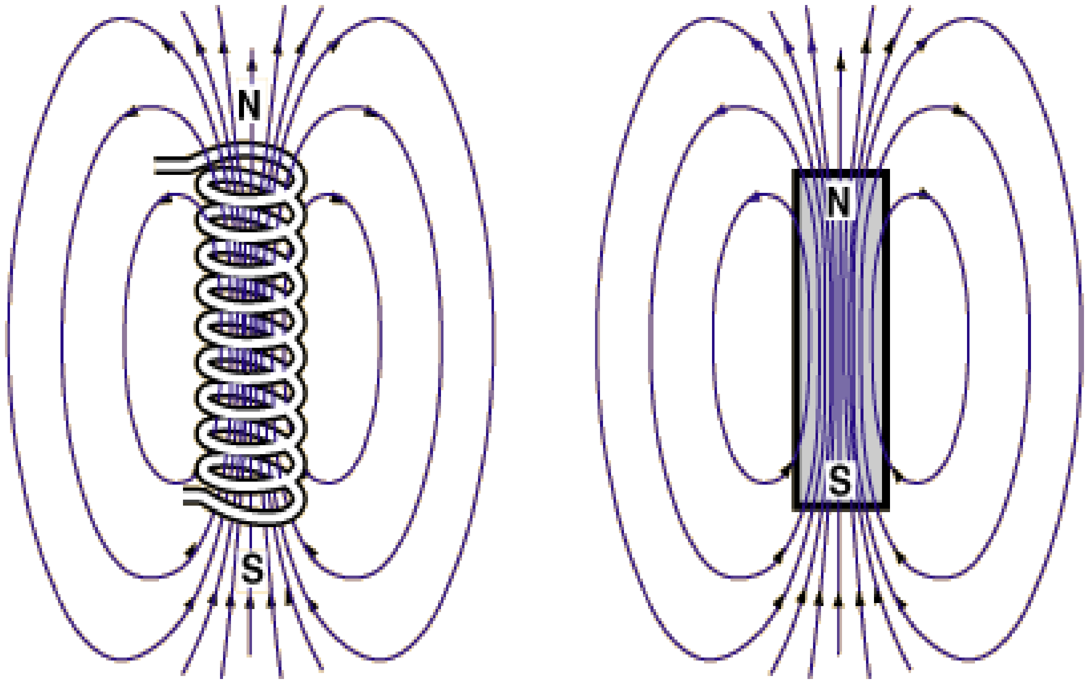
1. What is the source of magnetic field?
2. Which potential is used?
3. The governing equation in 2D symmetry.
4. Boundary conditions: Dirichlet, Neumann.
5. Calculated field Quantities: locals and integrals
6. Example: DC magnetic shielding.

Source of magnetic field

1. Current



2. Permanent Magnet



Starting from Maxwell's equations

	Integral Form	Differential Form	Conclusion
Gauss' theorem for magnetic field	Magnetic flux leaving any closed surface S is zero $\oint_S \mathbf{B} \cdot \mathbf{ds} = 0$	Magnetic field has no sources nor sinks $\mathrm{div}\mathbf{B} = 0$	No magnetic charge exists. It allows as introducing of vector magnetic potential **A**.
Ampere's Law	H-field circulation around a closed curve L is equal to free current through the surface S enclosed by L. $\oint_S \mathbf{H} \cdot \mathbf{dl} = I$	Magnetic field is not conservative in areas where the current exists. $\mathrm{rot}\mathbf{H} = \mathbf{J}$	Magnetic field is created by currents. Magnetic scalar potential Ψ only exists in current-free areas
Material Equation	$\mathbf{B} = \mu\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$		The state of material in magnetic field is determined by both external current and material magnetization

Which potential we use?

Electric Field

$\mathbf{rotE} = -\frac{\partial B}{\partial t} = 0$ Electrostatic field is conservative

Scalar potential φ exists: $\mathbf{E}$=-grad φ

Magnetic Field in current-free region

$\mathbf{rotH} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ Magnetic field is conservative in current-free area

Scalar magnetic potential ψ only exists in current-free areas: $\mathbf{H}$=-grad ψ

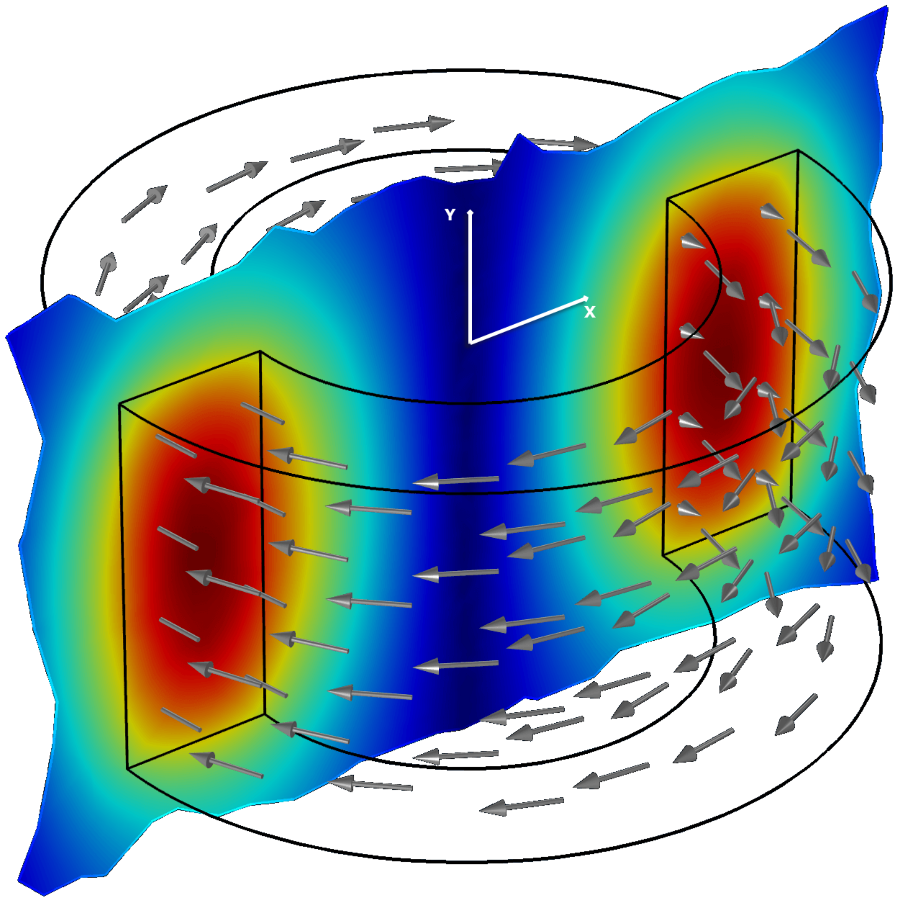
Magnetic Field everywhere

$\mathrm{div}\mathbf{B} = 0$ Magnetic field is solenoidal (divergent-free)

Vector magnetic potential $\mathbf{A}$ always exists: $\mathbf{B}$=rot$\mathbf{A}$

What happens with vector potential in 2D ?

Only out-of-plane component



$\mathbf{A}=\mathbf{i}_z A_z$

$\mathbf{J}=\mathbf{i}_z J_z$

$\mathbf{B}=\mathbf{i}_x B_x+\mathbf{i}_y B_y$

$\mathbf{H}=\mathbf{i}_x H_x+\mathbf{i}_y H_y$

Magnetic Flux

Stokes' theorem

$$\left\{\begin{array}{l} \Phi = \int_S \mathbf{B}\cdot\mathbf{ds} \\ \mathbf{B} = \mathrm{rot}\mathbf{A} \end{array}\right\} \Rightarrow \Phi = \int_S \mathrm{rot}\mathbf{A}\cdot\mathbf{ds} = \int_L \mathbf{A}\cdot\mathbf{dl}$$

Note!

$$\mathbf{B} = \mathrm{rot}\mathbf{A} \Rightarrow \left\{\begin{array}{l} B_x = \dfrac{\partial A}{\partial y} \\ B_y = -\dfrac{\partial A}{\partial x} \end{array}\right\}$$

Governing Equation of static magnetic field

$$\mathrm{rot}\left(\frac{1}{\mu}\mathrm{rot}\vec{\mathbf{A}}\right) = \vec{\mathbf{J}} + \mathrm{rot}\vec{\mathbf{H}}_{\mathbf{c}} \qquad \mu = f\left(\mathrm{rot}\vec{\mathbf{A}}\right)$$

Poisson equation
(coordinate independent form both 3D and 2D)

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu_y}\frac{\partial A}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mu_x}\frac{\partial A}{\partial y}\right) = -J + \left(\frac{\partial H_{cy}}{\partial x} - \frac{\partial H_{cx}}{\partial y}\right)$$

2D in Cartesian coordinates (x, y)
(plane-parallel symmetry)

$$\frac{\partial}{\partial r}\left(\frac{1}{r\mu_z}\frac{\partial (rA)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r\mu_r}\frac{\partial (rA)}{\partial z}\right) = -J + \left(\frac{\partial H_{cr}}{\partial z} - \frac{\partial H_{cz}}{\partial r}\right)$$

2D in Cylindrical coordinates (r, z)
(axial symmetry)

Magnetic boundary conditions

$$A = f(x, y)$$
$$rA = f(x, y)$$

Plane

Axisymmetric

Dirichlet Condition:
the known magnetic potential A (or rA)
(may depend on coordinates)

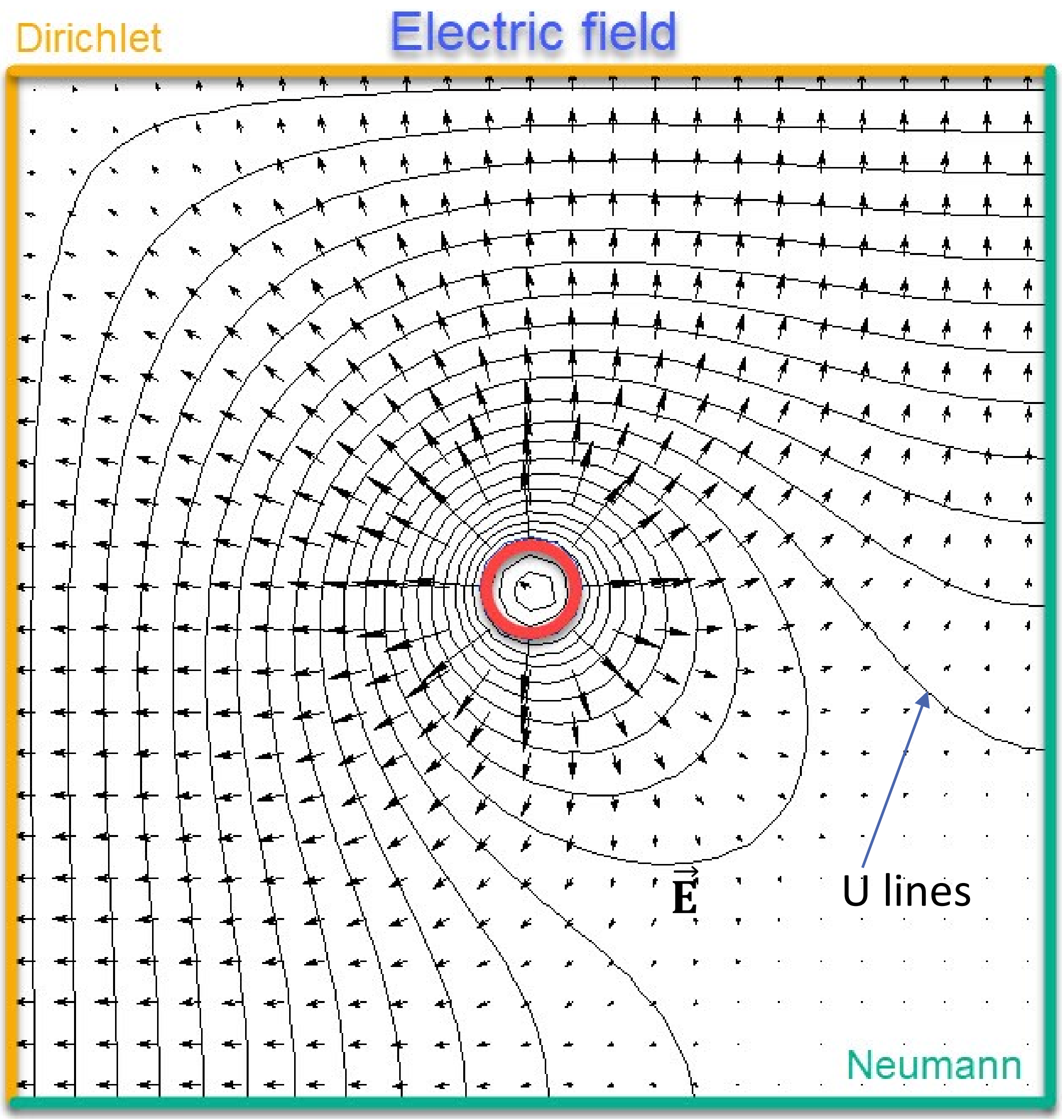
$$H_t = \frac{1}{\mu}\frac{\partial A}{\partial n} = \sigma(x, y)$$

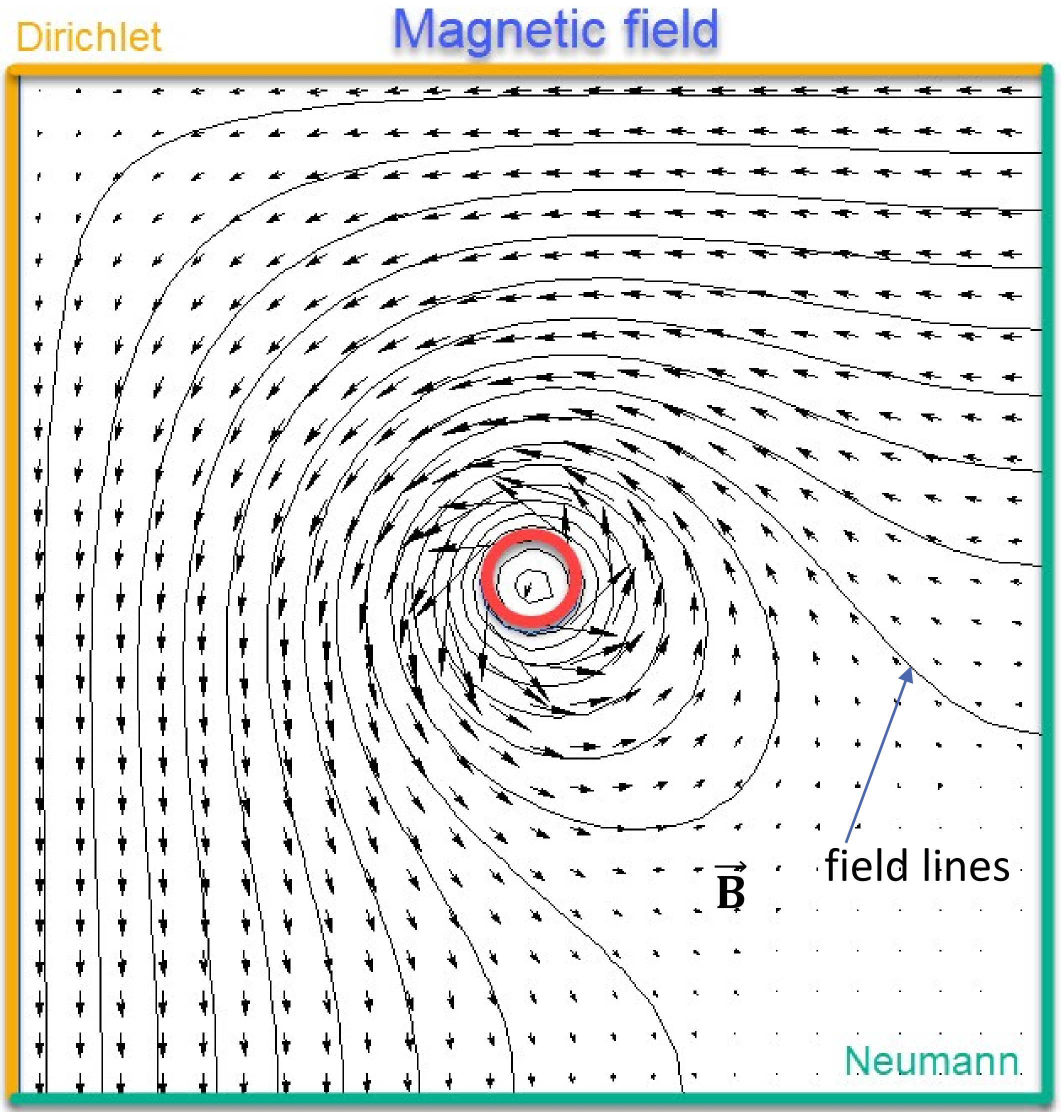
Neumann condition:
the known tangential H-field,
equal to surface current density

$$A = const \Leftrightarrow B_n = 0$$

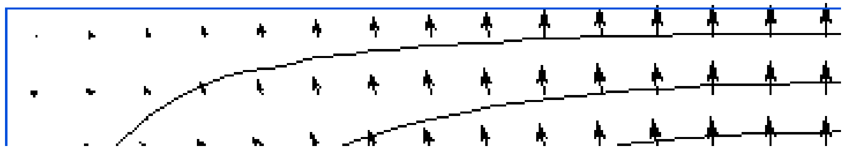
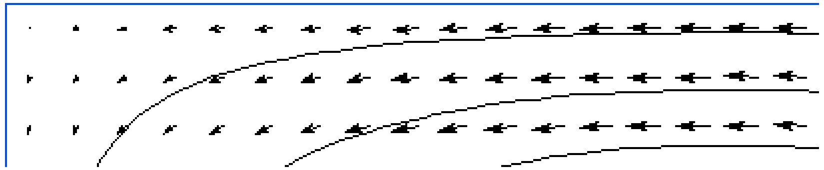
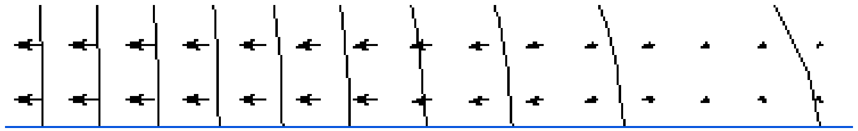
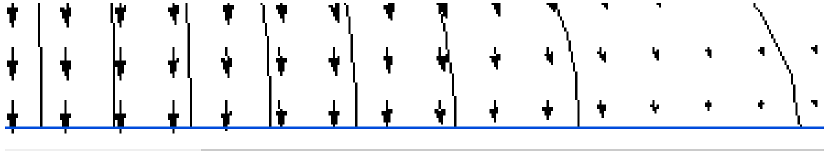
Zero normal flux (superconductor):
A special type of Dirichlet condition

Magnetic vs. electric boundary conditions





Magnetic vs. electric boundary conditions

	Electric field	Magnetic field
Dirichlet Condition:	Electric field lines are **normal** to a Dirichlet surface 	Magnetic field lines are **tangential** to a Dirichlet surface 
Neumann condition (zero):	Electric field lines are **tangential** to a Neumann surface 	Magnetic field lines are **normal** to a Neumann surface 

Calculation results

1. Locals

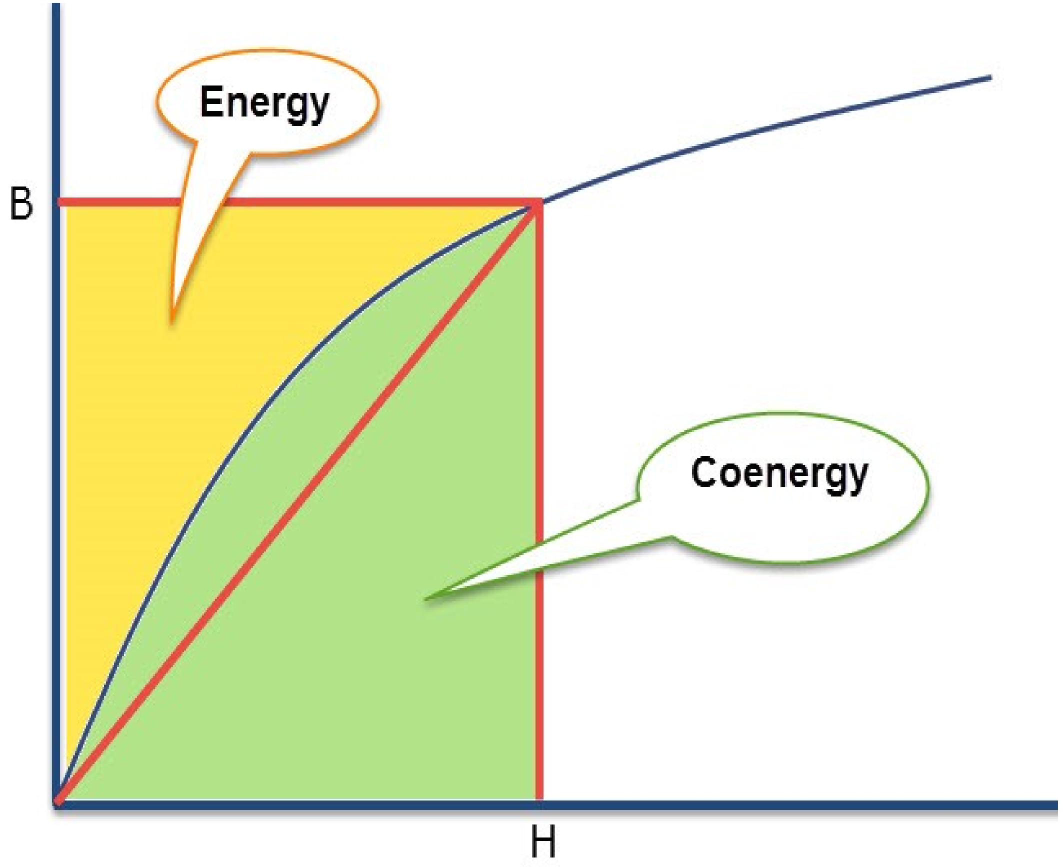
Magnetic flux density (B-Field): $\vec{\mathbf{B}} = \mathrm{rot}\vec{\mathbf{A}}$

Magnetic field strength (H-Field): $\vec{\mathbf{H}} = \frac{1}{\mu}\vec{\mathbf{B}} - \vec{\mathbf{H}}_c$

Magnetic energy density:

Linear material $w = \frac{\mathbf{B} \cdot \mathbf{H}}{2}$

Nonlinear material $w = \frac{1}{2}\int_0^B H(B')\,dB'$



Calculation results

2. Integrals

Mechanical force:

$$\vec{\mathbf{F}} = \frac{1}{2}\oint_S \big(\mathbf{H}(\mathbf{B}\cdot\mathbf{n}) + \mathbf{B}(\mathbf{H}\cdot\mathbf{n}) - \mathbf{n}(\mathbf{H}\cdot\mathbf{B})\big)\, ds$$

Mechanical torque:

$$\mathbf{T} = \frac{1}{2}\oint_S \big([\mathbf{r}\times\mathbf{H}](\mathbf{B}\cdot\mathbf{n}) + [\mathbf{r}\times\mathbf{B}](\mathbf{H}\cdot\mathbf{n}) - [\mathbf{r}\times\mathbf{n}](\mathbf{H}\cdot\mathbf{B})\big)\, ds$$

Magneto-motive force (MMF):

$$MMF = \int_L \mathbf{H}\cdot\mathbf{dl}$$

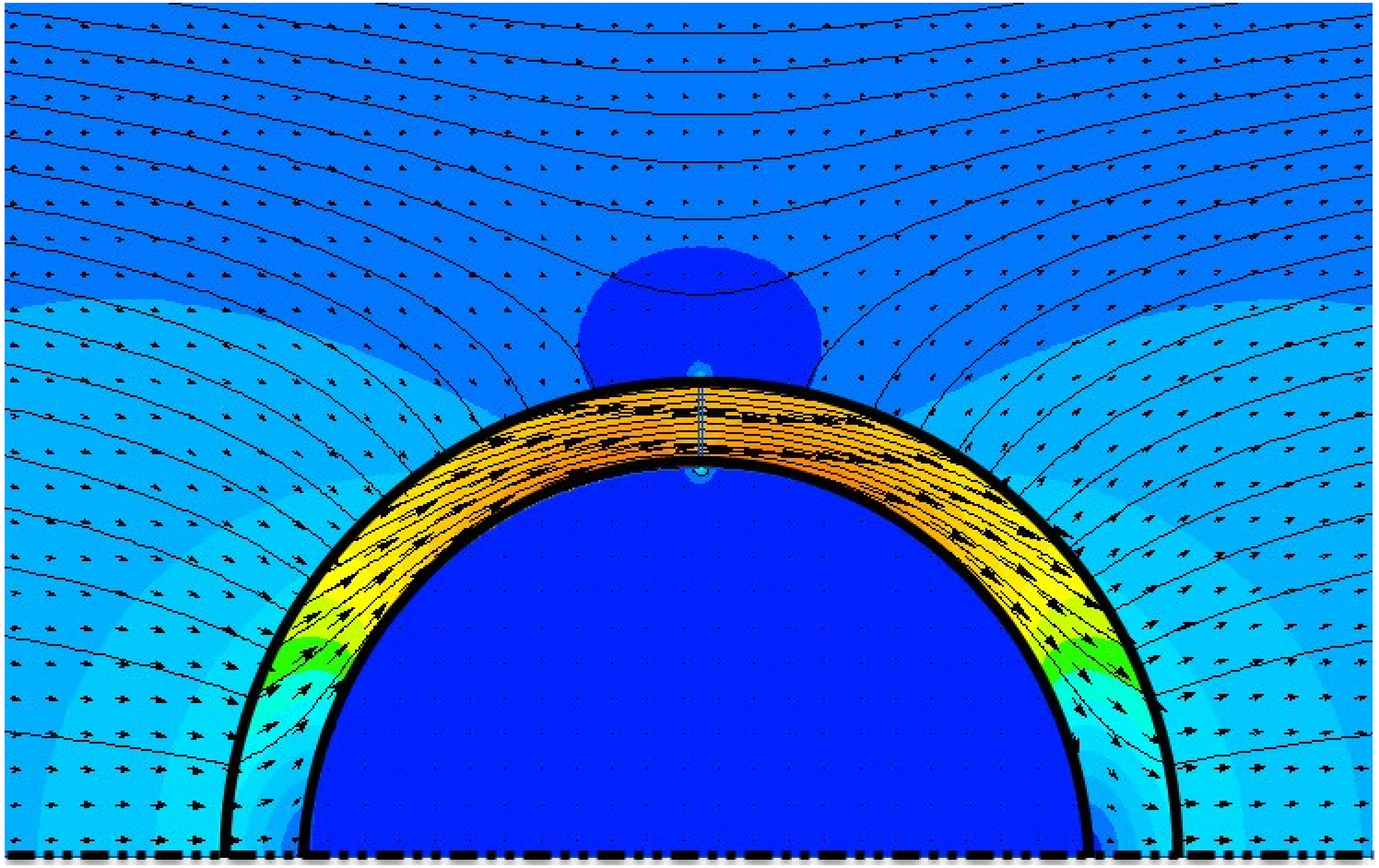
Flux Linkage:

$$\Psi = \frac{1}{S}\int_L A\cdot ds$$

Example: Lab 5 – DC Magnetic Shielding

Steel spherical shield
in uniform magnetic field B_0

B_0=12.6 mT



Sizing and material properties

Sizing:

External diameter	54 mm
Internal diameter	44 mm
Outer diameter	*500 mm*

B, T	H, A/m
0.030	50
0.206	150
0.325	200
0.445	250
0.535	300
0.600	350
0.650	400
0.685	450
0.720	500
0.745	550
0.770	600
0.800	650
0.820	700
0.840	750
0.860	800

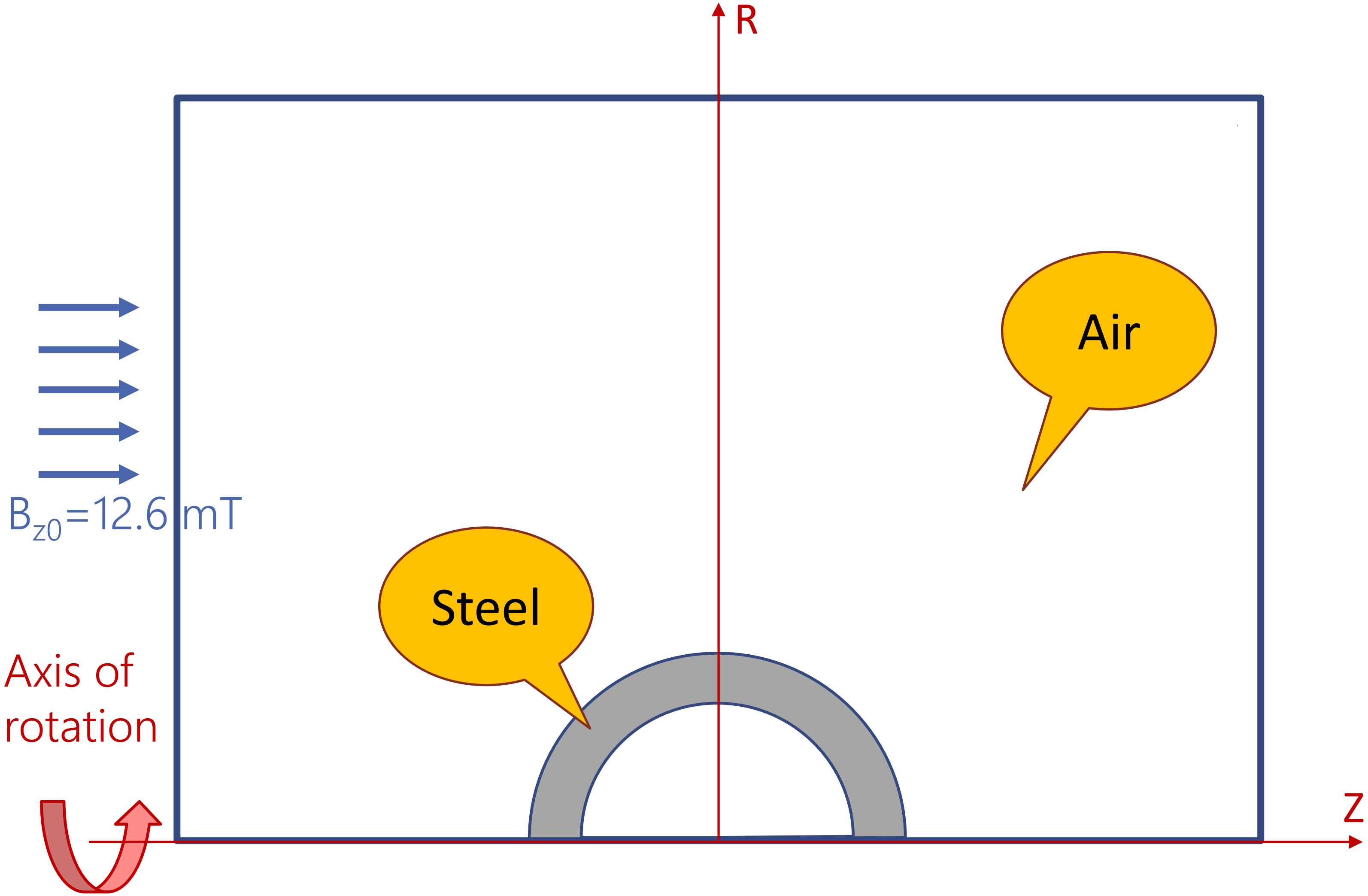
Z-uniform field boundary condition

$$\mathbf{B} = \mathrm{rot}\mathbf{A} = \begin{Bmatrix} B_Z = \frac{1}{r}\frac{\partial(rA)}{\partial r} \\ B_R = -\frac{1}{r}\frac{\partial(rA)}{\partial z} \end{Bmatrix}$$

Choose as: $rA = 0.5r^2 B_{Z0}$

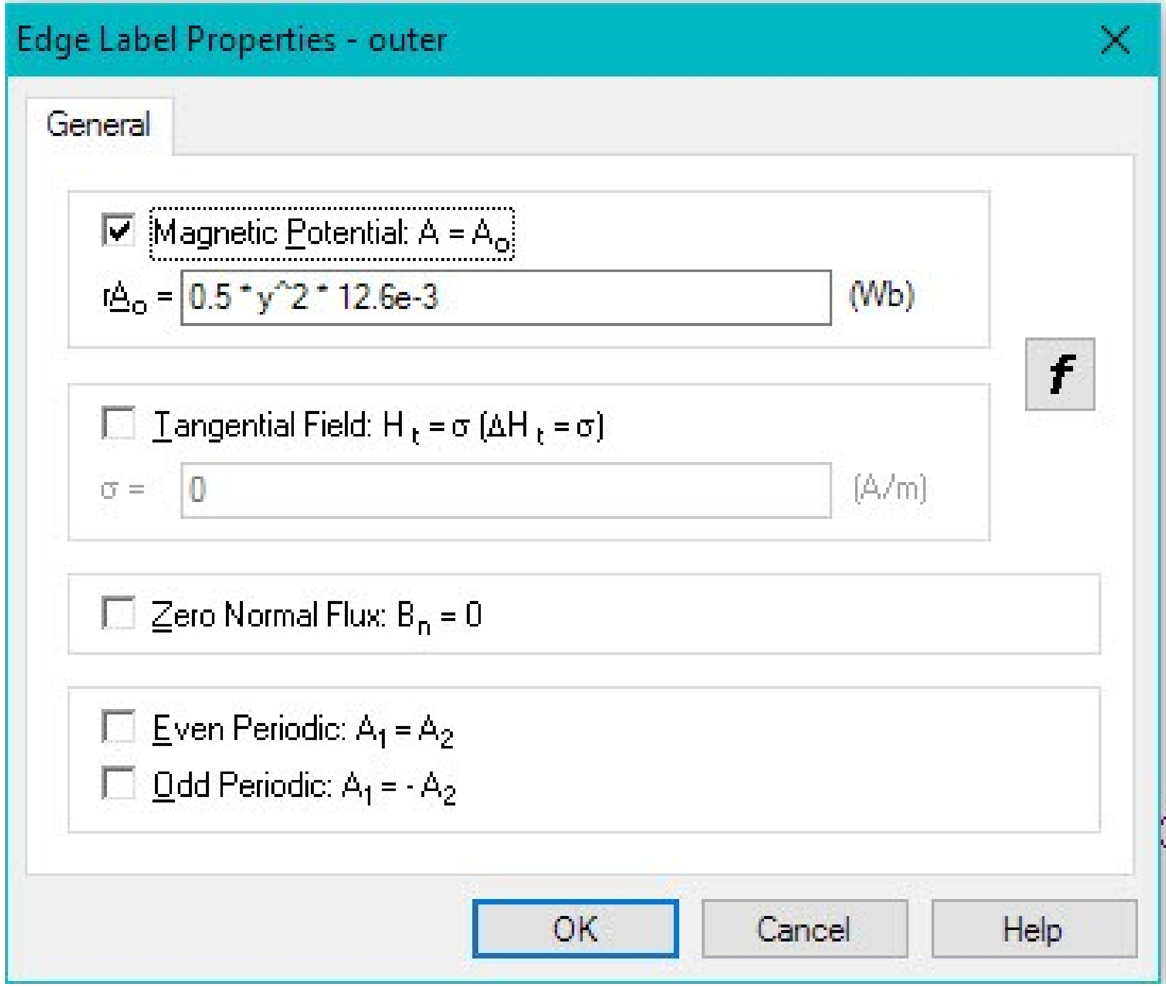
Check it by differentiating:

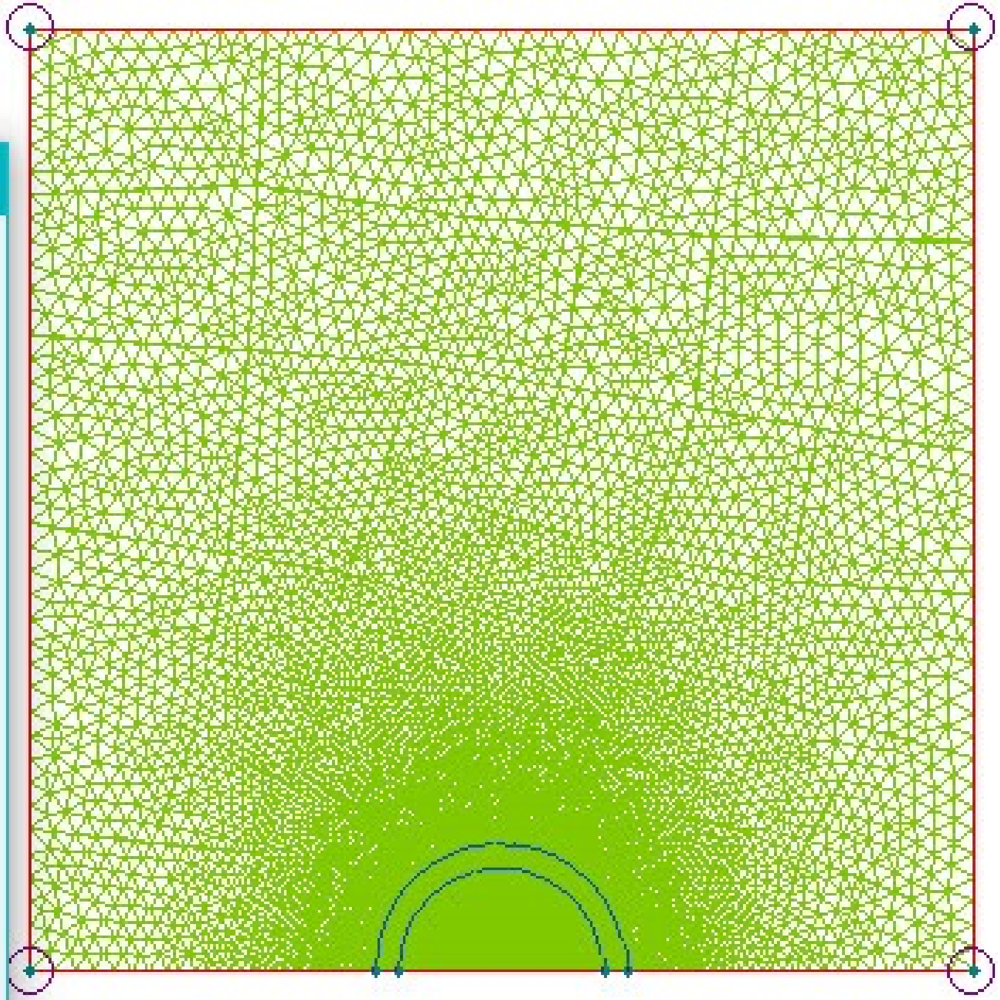
$$B_Z = \frac{1}{r}\frac{\partial\left(\frac{1}{2}r^2 B_{Z0}\right)}{\partial r} = B_{Z0}$$



Z-uniform field boundary condition

Dirichlet boundary condition: $rA = 0.5r^2B_{Z0}$





Shielding Factor

Field outside $B_0 = 12.6\ mT$

Air gap across the field

Field inside shield $B_i = 0.807\ mT$

The shielding factor:

$$k = 12.6/0.807 = 15.6$$

Air gap along the field

Field inside shield $B_i = 0.158\ mT$

The shielding factor:

$$k = 12.6/0.158 = 79.7$$